A Tight Linear Time (1/2)-Approximation for Unconstrained Submodular Maximization

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Combinatorial Optimization Problems

Maximization or minimization problems in which, for each input there is a set of feasible solutions and, for each solution there is a cost associated with it.

In this presentation we will focus on the Unconstrained Submodular Maximization problem (USM).

This is one of the most basic submodular optimization problems, that captures some well known problems as Max-Cut and Max-DiCut.

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Unconstrained Submodular Maximization

It is a maximization problem in which we are given a non-negative submodular function $f : 2^N \to \mathbb{R}^+$.

The objective is to find a subset $S \subseteq N$ maximizing $f(S)$.

Problems with submodular objective functions capture the principle of economy of scale, and are commonly used in economics and algorithmic game theory.

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Submodular Functions

A function is submodular if, for every $A \subseteq B \subseteq N$ and $u \in N$, we have:

$$f(A \cup \{u\}) - f(A) \geq f(B \cup \{u\}) - f(B).$$

An equivalent definition is, for any subsets A and B:

$$f(A) + f(B) \geq f(A \cup B) + f(A \cap B).$$

As an example, consider the cardinality of a cut in a graph.

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Paper Results

They design a linear time deterministic (1/3)-approximation algorithm for USM, using a greedy based approach.

Then, modifying the deterministic algorithm using randomness, they design a (1/2)-approximation algorithm for USM.

This result is tight, because there is an upper bound of $(1/2 + \epsilon)$ to the approximation ratio of any algorithm for USM [2].

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Techniques

Lets show two straightforward greedy approachs.

First, define $\bar{f}(S) = f(N \setminus S)$.

Once $f(S)$ is submodular so it is $\bar{f}(S)$.

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\begin{aligned}
\bar{f}(A) + \bar{f}(B) &= f(N \setminus A) + f(N \setminus B) \\
&\geq f((N \setminus A) \cup (N \setminus B)) + f((N \setminus A) \cap (N \setminus B)) \\
&= f(N \setminus (A \cap B)) + f(N \setminus (A \cup B)) \\
&= \bar{f}(A \cap B) + \bar{f}(A \cup B).
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\end{aligned}$$

Techniques (cont.)

Now, lets define a greedy algorithm that starts from an empty solution and iteratively adds elements to it.

This algorithm decides to add an element by checking if the submodular function increases when it is added.

It works both for f and $\bar{f}$, and for the later it corresponds to start with N and to iteratively remove elements from it.

Although they seem reasonable, neither gives a constant approximation ratio.

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Deterministic Algorithm

Algorithm 1: DeterministicUSM.

Data: f, N

$X_0 \leftarrow \emptyset$; $Y_0 \leftarrow N$;

for $i = 1$ *to* $|N|$ **do**

  $a_i \leftarrow f(X_{i-1} \cup \{u_i\}) - f(X_{i-1})$;

  $b_i \leftarrow f(Y_{i-1} \setminus \{u_i\}) - f(Y_{i-1})$;

  if $a_i \geq b_i$ **then**

    $X_i \leftarrow X_{i-1} \cup \{u_i\}$; $Y_i \leftarrow Y_{i-1}$;

  else $a_i < b_i$

    $X_i \leftarrow X_{i-1}$; $Y_i \leftarrow Y_{i-1} \setminus \{u_i\}$;

  end

end

return X_n (or equivalently Y_n).

Analysis of the DeterministicUSM Algorithm

Lemma (1)

For every $1 \leq i \leq |N|$ *we have that* $a_i + b_i \geq 0$.

Demonstração.

By submodularity, we have:

$$f(X_{i-1} \cup \{u_i\}) - f(X_{i-1}) \geq f(Y_{i-1}) - f(Y_{i-1} \setminus \{u_i\}).$$

So:

$$\begin{aligned} a_i + b_i &= f(X_{i-1} \cup \{u_i\}) - f(X_{i-1}) + f(Y_{i-1} \setminus \{u_i\}) - f(Y_{i-1}) \\ &= (f(X_{i-1} \cup \{u_i\}) - f(X_{i-1})) - (f(Y_{i-1}) - f(Y_{i-1} \setminus \{u_i\})) \\ &\geq 0. \end{aligned}$$

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Analysis of the DeterministicUSM (cont.)

Lets define $OPT_i = (OPT \cup X_i) \cap Y_i$.

Realize that $OPT_0 = OPT$ and $OPT_{|N|} = X_{|N|} = Y_{|N|}$.

Lemma (2)

For every $1 \leq i \leq |N|$ we have:

$$f(OPT_{i-1}) - f(OPT_i) \leq f(X_i) - f(X_{i-1}) + f(Y_i) - f(Y_{i-1}).$$

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Analysis of the DeterministicUSM (cont.)

Theorem

The DeterministicUSM algorithm is a linear time (1/3)-approximation for USM.

Demonstração.

Using lemma 2 we have:

$$\sum_{i=1}^{|N|}(f(OPT_{i-1}) - f(OPT_i)) \leq \sum_{i=1}^{|N|}(f(X_i) - f(X_{i-1})) + \sum_{i=1}^{|N|}(f(Y_i) - f(Y_{i-1})).$$

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Analysis of the DeterministicUSM (cont.)

Proving theorem (cont).

Demonstração.

Once the previous sums are telescopic we have:

$$\begin{aligned} f(OPT_0) - f(OPT_{|N|}) &\leq f(X_{|N|}) - f(X_0) + f(Y_{|N|}) - f(Y_0) \\ &\leq f(X_{|N|}) + f(Y_{|N|}). \end{aligned}$$

So,

$$f(OPT) \leq 3f(X_{|N|}).$$

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Proving lemma 2.

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Assume that $a_i \geq b_i$ (the other case is similar).

In this case, $OPT_i = (OPT \cup X_i) \cap Y_i = OPT_{i-1} \cup \{u_i\}$ and $Y_i = Y_{i-1}$.

So, we have to prove that:

$$f(OPT_{i-1}) - f(OPT_{i-1} \cup \{u_i\}) \leq f(X_i) - f(X_{i-1}) = a_i.$$

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Analysis of the DeterministicUSM (cont.)

Proving lemma 2 (cont).

Demonstração.

Now we consider two cases.

If $u_i \in OPT$ then $f(OPT_{i-1}) - f(OPT_{i-1}) = 0$ and $a_i \geq 0$.

If $u_i \notin OPT$ then $u_i \notin OPT_{i-1}$ and

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Thank you!

Questions?

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