The Online Connected Facility Location Problem[2]

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April 3rd, 2014

[2]grant No. 2012/06728-3, São Paulo Research Foundation (FAPESP)

Combinatorial Optimization Problems

- Maximization or minimization problems,
- Algorithm receives an input,
- Returns a solution with a cost.

Some minimization problems in which we are interested are:

- Facility Location problem,
- Steiner Tree problem,
- Connected Facility Location problem.

These problems are NP-hard with constant factor approximation algorithms known.

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- Defined in a metric space with distance and facility costs,
- Algorithm receives a set of clients and connects each client to an opened facility,
- Goal: minimize total cost of opening facilities plus connecting clients.

$$\sum_{i \in F'} f(i) + \sum_{j \in D} d(j, F')$$

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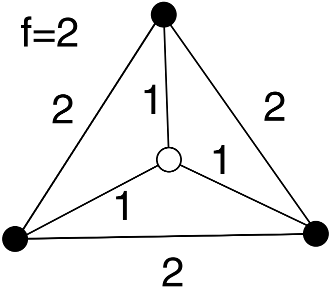

Total cost = 2 + 3 = 5.

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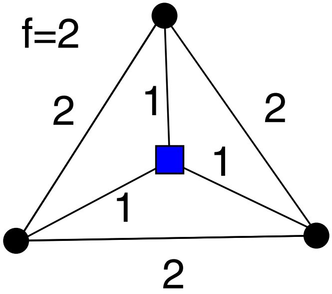

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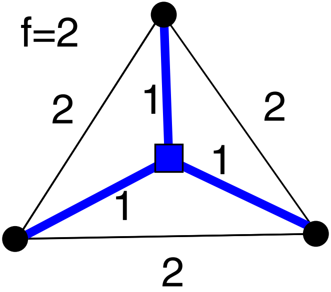

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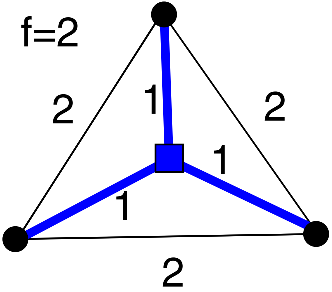

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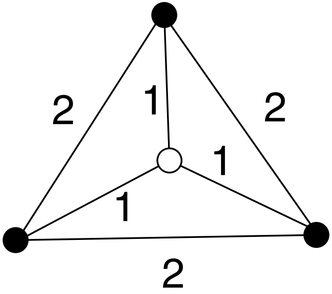

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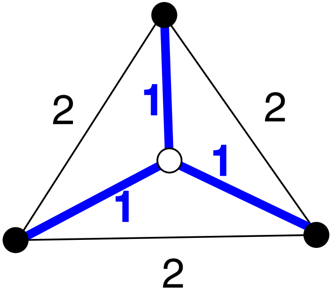

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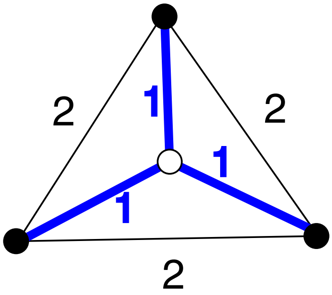

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The Connected Facility Location Problem

- Combination of Facility Location and Steiner Tree.
- Motivation is to build a two level network.
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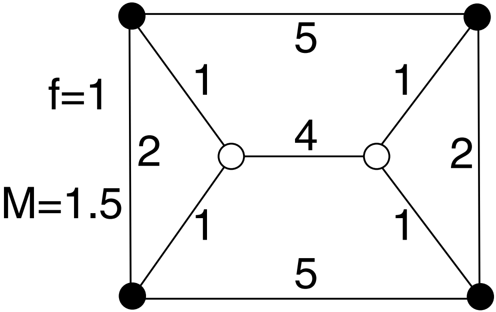

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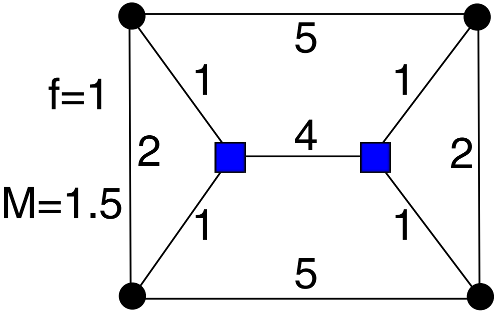

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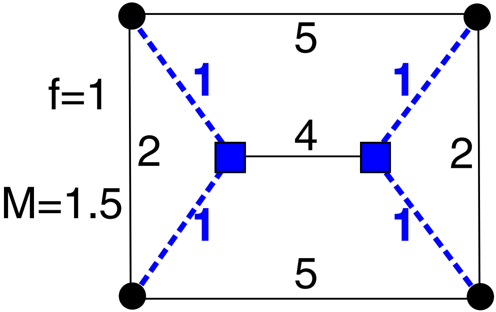

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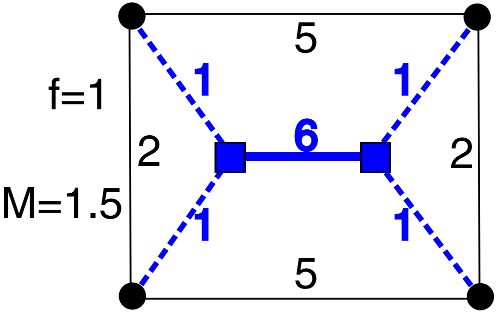

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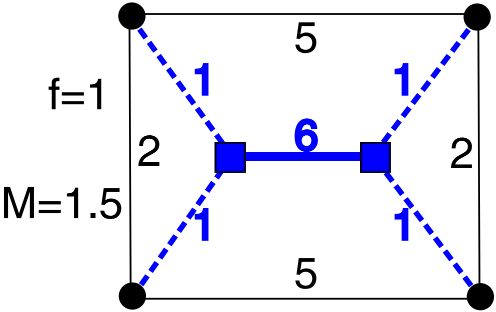

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Online Problems

- Parts of the input are revealed one at a time,
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- Technique used to analyse online algorithms,
- An online algorithm ALG is c-competitive if:

$$\mathrm{ALG}(I) \leq c\mathrm{OPT}(I) + \alpha,$$

  for every input I and some α constant.
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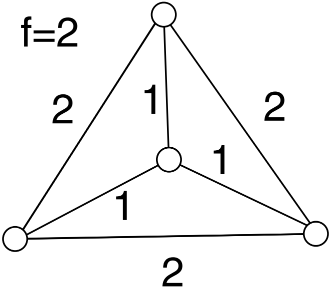

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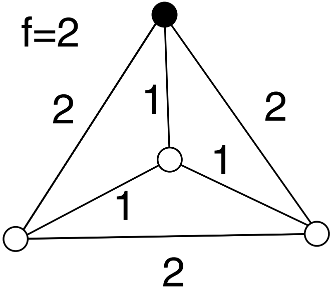

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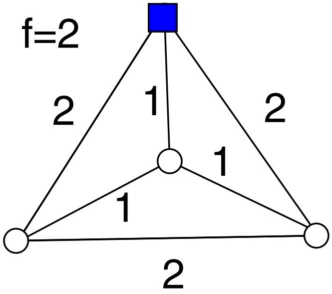

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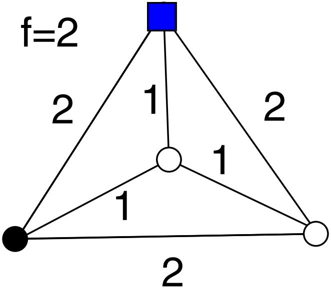

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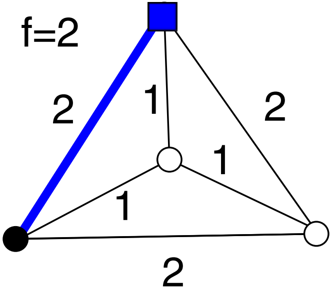

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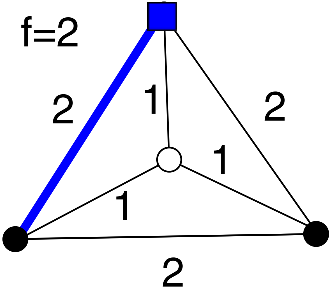

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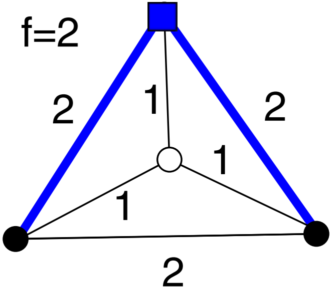

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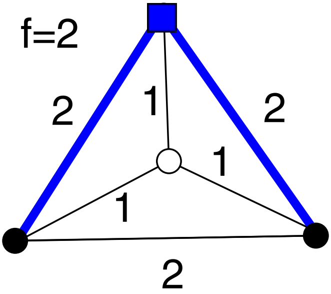

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The Online Facility Location Problem (cont.)

- There are $O(\log n)$-competitive algorithms known,
- In particular, a primal-dual algorithm due to Fotakis that is used in our result,
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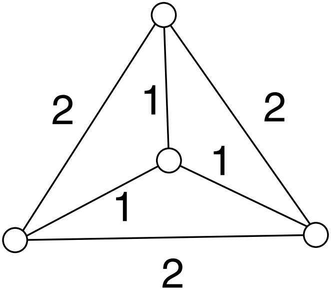

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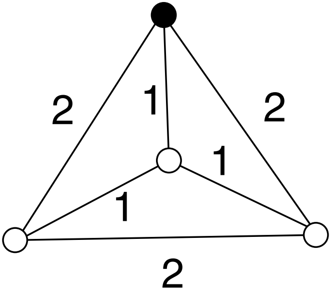

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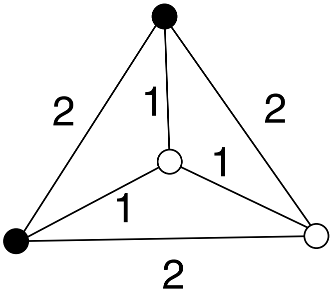

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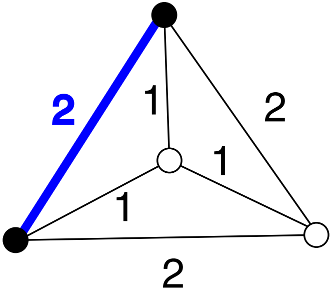

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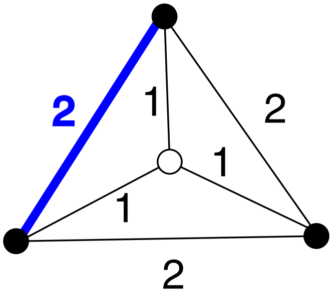

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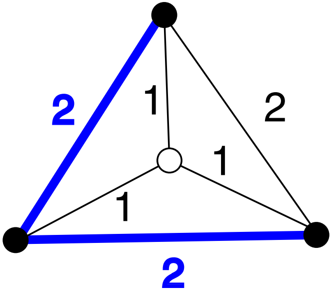

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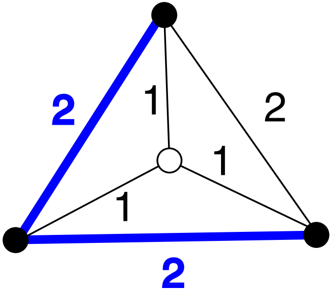

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Online Connected Facility Location Problem

- Combination of Online Facility Location and Online Steiner Tree, is similar to the Connected Facility Location,
- Clients arrive one at a time and each one must be immediately connected to some facility,
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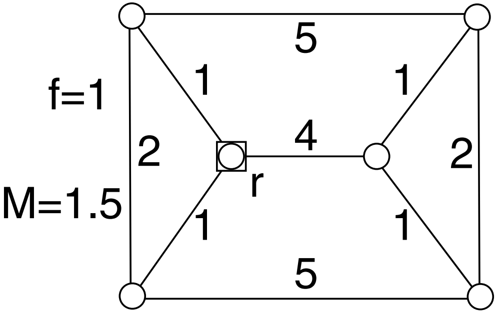

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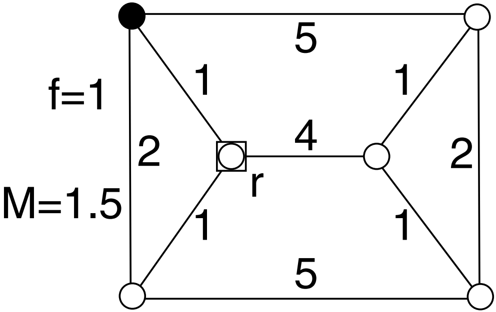

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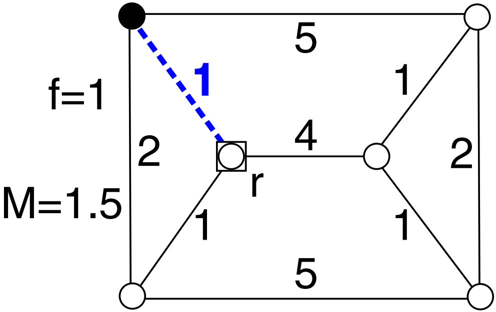

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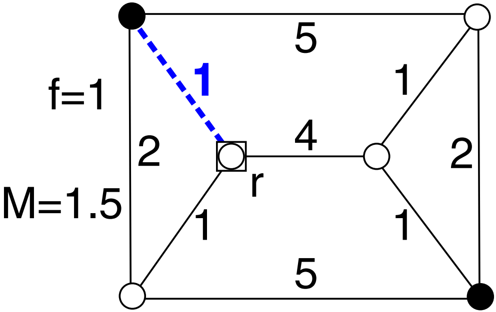

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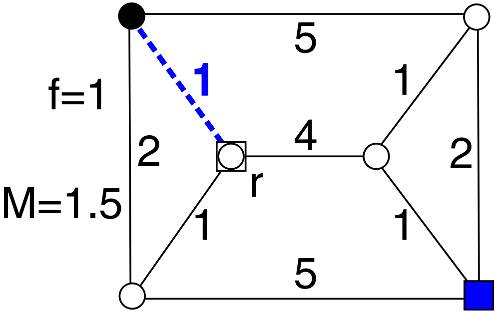

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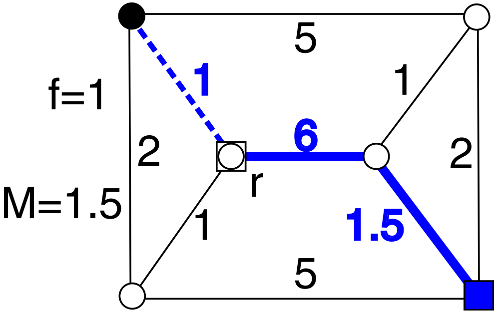

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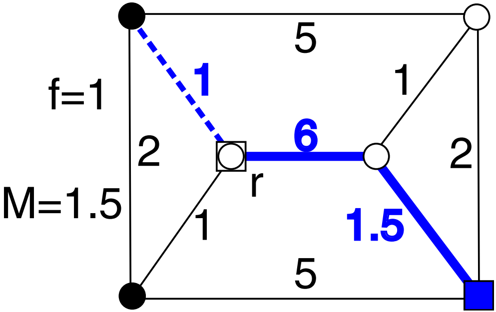

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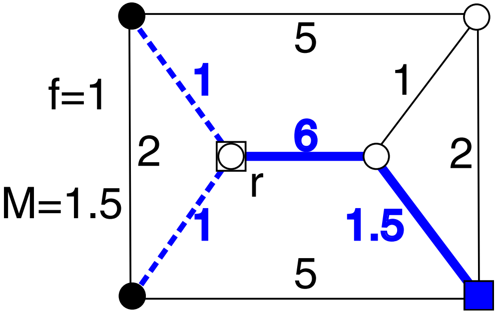

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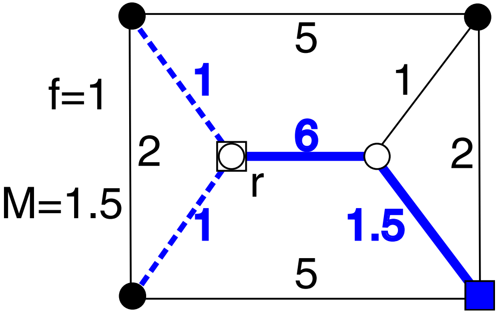

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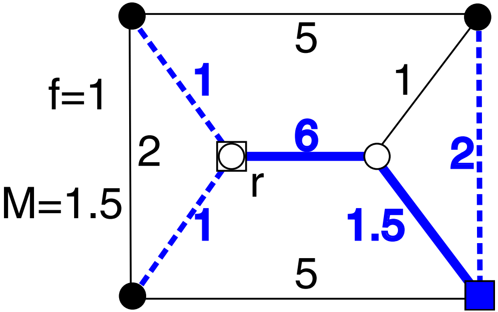

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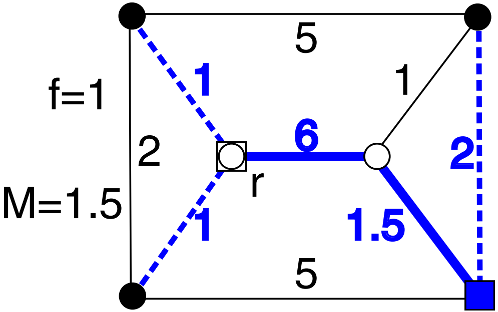

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Online CFL Algorithm

- Sample-and-augment algorithm based on algorithm for CFL due to Eisenbrand et al.
- Sample-and-Augment is a randomized technique, due to Gupta et al., to design algorithms for rent-or-buy problems.
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Algorithm 1: The Online CFL algorithm.

Data: $G = (V, E)$, d, f, F, root r and M

Initializing auxiliar sets and compFL;

while *a new client j arrives* **do**

  send j to compFL; /* Update virtual OFL solution */

  mark j with probability $p = \frac{1}{M}$;

  if *j is marked and connected to a facility i that is not open* **then**

    $F' \leftarrow F' \cup \{i\}$; /* Open new facility */

    $T \leftarrow T \cup \{(i, j)\} \cup \{path(j, V(T))\}$;

    /* Connect new facility to the tree */

  end

  let i be the closest open facility to j;

  $D \leftarrow D \cup \{j\}$; $a(j) \leftarrow i$; /* Connect the client */

end

return $(F' \setminus \{r\}, T, a)$;

Analysis of the Online CFL Algorithm

Algorithm cost is divided between facilities opening cost (O), clients connection cost (C) and Steiner tree cost (S):

$$ALG_{OCFL}(D) = O + C + S.$$

The cost of the offline optimal solution is also divided in this way:

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Analysis of the Online CFL Algorithm (cont.)

Lemma (Opening Cost)

$O \leq c_{\mathrm{OFL}}(O^* + C^*)$.

- Let O_{compFL} be the facility opening cost of compFL,
- Our algorithm opens a subset of the facilities opened by compFL,
- An optimal solution for CFL is a feasible solution for FL,
- So we have that:

$$O \leq O_{\mathrm{compFL}} \leq c_{\mathrm{OFL}} \mathrm{OPT}_{\mathrm{FL}}(D) \leq c_{\mathrm{OFL}}(O^* + C^*) .$$

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$E[S] \leq c_{\mathrm{OST}}(S^* + C^*) + c_{\mathrm{OFL}}(O^* + C^*)$.

- Our algorithm builds a tree connecting only marked clients (D'),
- And augments it connecting each client j to facility $a(j)$,
- The idea is to bound these costs by OPT_{ST} and OPT_{FL}.

$$\begin{aligned} E[S] &\leq E[M\mathrm{compST}(D')] + E\left[M \sum_{j \in D'} d(j, a(j))\right] \\ &\leq E[M c_{\mathrm{OST}} \mathrm{OPT}_{\mathrm{ST}}(D')] + c_{\mathrm{OFL}} \mathrm{OPT}_{\mathrm{FL}}(D) \\ &\leq c_{\mathrm{OST}}(S^* + C^*) + c_{\mathrm{OFL}}(O^* + C^*) \ . \end{aligned}$$

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Lemma (Connection Cost)

$E[C] \leq c_{\mathrm{OFL}}(O^* + C^*) + c_{\mathrm{OST}}(S^* + C^* + c_{\mathrm{OFL}}(O^* + C^*)).$

- This proof uses similar ideas to the previous one,
- Together with a conditional probability argument,
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$E[\mathrm{ALG}_{\mathrm{OCFL}}(D)] \in \mathrm{O}(\log^2 n \mathrm{OPT}_{\mathrm{CFL}}(D))$.

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Acknowledgements

Thank you!

Questions?

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