Online Combinatorial Optimization Problems

Mário César San Felice

Postdoc at Institute of Mathematics and Statistics - University of São Paulo

Seminar series at Department of Computing - Federal University of São Carlos

November 29th, 2017

Combinatorial Optimization Problems

Maximization or minimization problems

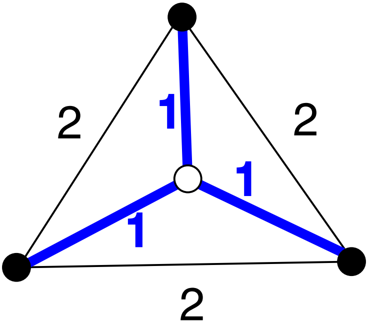
Algorithm receives an input

Returns a solution with a cost

As an example, lets take the Steiner Tree Problem

Steiner Tree Problem

Input: $G = (V, E)$, $d : E \to \mathbb{R}^+$, terminals $D \subseteq V$



$$\min \sum_{e \in T} d(e)$$

Total cost $= 3$

Online Problems

Input parts arrive one at a time

Each part is served before next one arrives

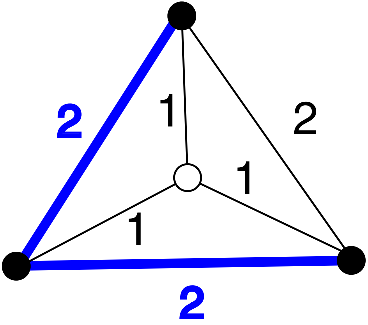
No decision can be changed in the future

As an example, lets take the Online Steiner Tree problem

Online Steiner Tree Problem

Terminal nodes arrive one at a time

No edge used can be removed in the future



$$\min \sum_{e \in T} d(e)$$

Total cost $= 2 + 2 = 4$

Competitive Analysis

Worst case analysis technique

For online algorithm ALG

Using offline optimal solution OPT

ALG is c-competitive if

$$\mathrm{ALG}(I) \leq c\,\mathrm{OPT}(I)$$

for every input I

As an example, lets take a greedy online algorithm for the Online Steiner Tree problem

Greedy Online Steiner Tree Algorithm

Algorithm 1: OST Algorithm

Input: (G, d)
$T \leftarrow (\emptyset, \emptyset)$;
while *a new terminal j arrives* **do**
  $T \leftarrow T \cup \{\mathrm{path}(j, V(T))\}$;
return T;

This algorithm is $\mathrm{O}(\log n)$-competitive [Imase and Waxman 1991]

A $\Omega(\log n)$ lower bound is known [Imase and Waxman 1991]

Areas of Interest

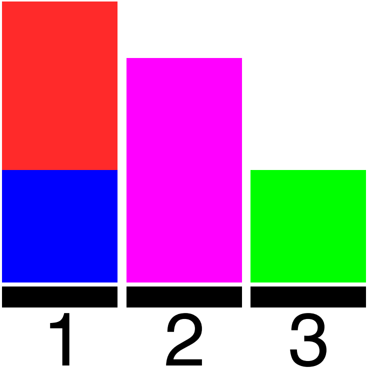
Online problems capture uncertainty over time

Common in operations research and computer science:

- Resource management: scheduling, packing and load balancing problems
- Dynamic data structures: list access problem
- Memory management: paging problem
- Sustainability: ski-rental problem
- Network design: Steiner tree and facility location problems

Online Load Balancing problem

Input: machines M, tasks D, sizes $s : D \to \mathbb{R}^+$



$$l(i) = \sum_{j \in D : a(j) = i} s(j) \qquad \min \max_{i=1}^{M} l(i)$$

Algorithm 2: OLB Algorithm

Input: M

For each machine $i = 1, \ldots, M$ set its load $l(i)$ to 0;

$i^* \leftarrow 1$;

while *a new task j arrives* **do**

  $a(j) \leftarrow i^*$;

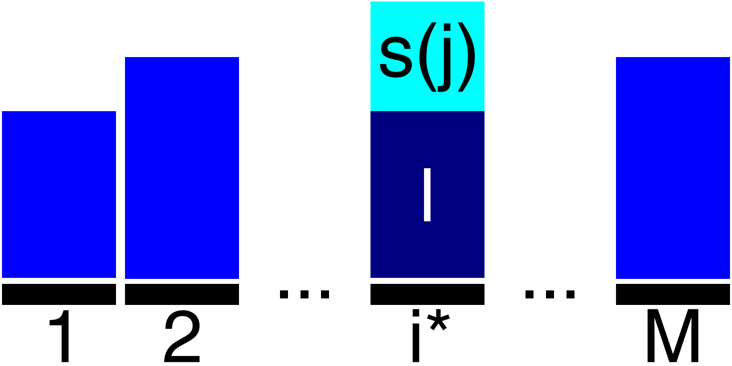
  $l(i^*) \leftarrow l(i^*) + s(j)$;

  choose machine with minimum load as new i^*;

return a;

OLB Algorithm is $(2 - \frac{1}{M})$-competitive

Let i^* be the machine with maximum load, j be the last task assigned to i^*, and $l(i^*) = l + s(j)$



We have $\mathrm{OPT} \geq s(j)$ and $\mathrm{OPT} \geq l + \frac{s(j)}{M}$

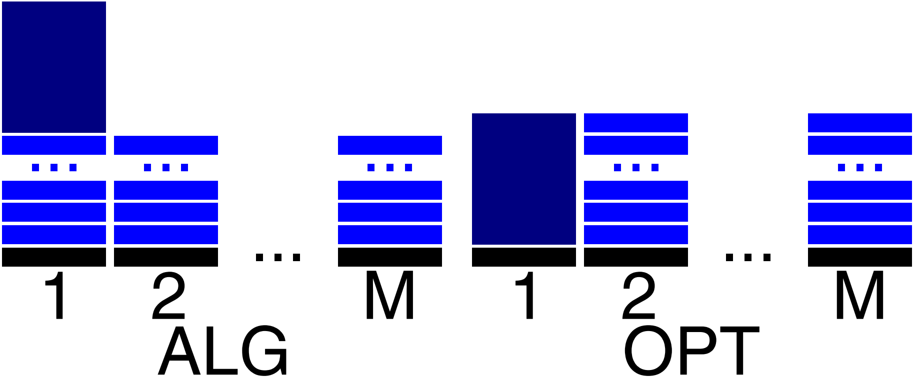
OLB Algorithm is $\left(2 - \frac{1}{M}\right)$-competitive

Since $\mathrm{OPT} \geq s(j)$ and $\mathrm{OPT} \geq l + \frac{s(j)}{M}$, we have

$$\begin{aligned}
\mathrm{ALG} &= l + s(j) \\
&\leq \mathrm{OPT} - \frac{s(j)}{M} + s(j) \\
&\leq \mathrm{OPT} + \left(1 - \frac{1}{M}\right)\mathrm{OPT} \\
&= \left(2 - \frac{1}{M}\right)\mathrm{OPT}
\end{aligned}$$

Lower Bound for OLB Algorithm

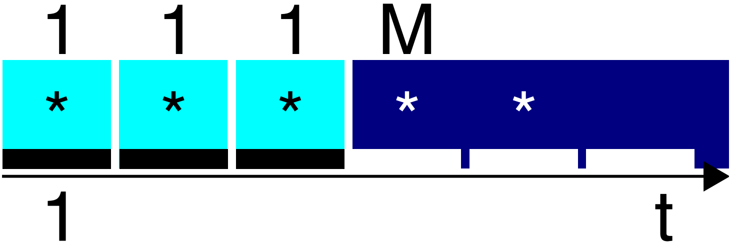
List with $M(M-1)$ size 1 tasks followed by one size M task



We have $\mathrm{ALG} = 2M - 1$ and $\mathrm{OPT} = M$

Ski Rental Problem

Input: time horizon, skis buying price M (renting cost is 1 per day), list informing when snow melts



minimize sum of renting days plus M (if we decide to buy skis)

Does a greedy algorithm solve this problem?

Ski Rental Application and Generalization

Ski rental algorithms are useful to save energy

Help to decide when to turn off parts of a system

Like cores in a processor or computers in a cluster

Generalized into Parking Permit Problem [Meyerson 2005]

Important both to theoretical and practical leasing problems

Ski Rental Algorithm

Algorithm 3: Intuitive SR Algorithm

Input: M
Set day j and total renting cost r to 0;
while *a new snow day happens* **do**
    if $r + 1 < M$ **then**
        Rent skis at day j and $r \leftarrow r + 1$;
    else
        Buy skis if still don't have them;
    $j \leftarrow j + 1$;

This algorithm is 2-competitive. Why?

Ski Rental LP Formulations

Linear programming relaxation

$$\begin{aligned} \min \quad & Mx + \textstyle\sum_{j=1}^{n} y_j & \\ \text{s.t.} \quad & x + y_j \geq 1 & \text{for } j = 1, \ldots, n \\ & x \geq 0, y_j \geq 0 & \text{for } j = 1, \ldots, n \end{aligned}$$

and its dual

$$\begin{aligned} \max \quad & \textstyle\sum_{j=1}^{n} \alpha_j & \\ \text{s.t.} \quad & \textstyle\sum_{j=1}^{n} \alpha_j \leq M & \\ & \alpha_j \leq 1 & \text{for } j = 1, \ldots, n \\ & \alpha_j \geq 0 & \text{for } j = 1, \ldots, n \end{aligned}$$

Primal-Dual Ski Rental Algorithm

Algorithm 4: Primal-Dual SR Algorithm

Input: M

Set day j' to 0;

while *a new snow day happens* **do**

increase $\alpha_{j'}$ until one of the following happens:

(a) $\alpha_{j'} = 1$; /* rent skis setting $y_{j'} = 1$ */

(b) $M = \alpha_{j'} + \sum_{j=1}^{j'-1} \alpha_j$; /* buy skis setting $x = 1$ */

$j' \leftarrow j' + 1$;

Is it similar to the previous algorithm?

Primal-Dual SR Algorithm is 2-Competitive

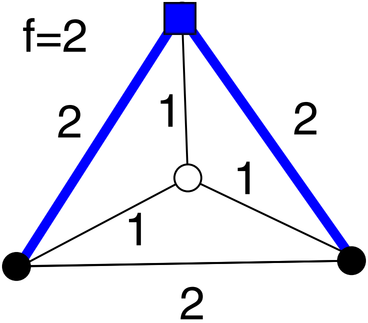
Cost of any dual solution is at most OPT

So

$$
\begin{aligned}
ALG &= Mx + \sum_{j=1}^{n} y_j \\
&\leq \sum_{j=1}^{n} \alpha_j + \sum_{j=1}^{n} \alpha_j \\
&\leq 2\mathrm{OPT}
\end{aligned}
$$

Online Facility Location Problem

Input: $G = (V, E)$, $d : E \to \mathbb{R}^+$, $f : V \to \mathbb{R}^+$, clients $D \subseteq V$



$$\min \sum_{i \in F^a} f(i) + \sum_{j \in D} d(j, a(j))$$

Total cost $= 2 + 2 + 2 = 6$.

Online Facility Location LP Formulation

Linear programming relaxation

$$\begin{array}{rll} \min & \sum_{i \in F} f(i) y_i + \sum_{j \in D} \sum_{i \in F} d(j,i) x_{ji} & \\ \text{s.t.} & x_{ji} \le y_i & \text{for } j \in D \text{ and } i \in F \\ & \sum_{i \in F} x_{ji} \ge 1 & \text{for } j \in D \\ & y_i \ge 0, x_{ji} \ge 0 & \text{for } j \in D \text{ and } i \in F \end{array}$$

and its dual

$$\begin{array}{rll} \max & \sum_{j \in D} \alpha_j & \\ \text{s.t.} & \sum_{j \in D} (\alpha_j - d(j,i))^+ \le f(i) & \text{for } i \in F \\ & \alpha_j \ge 0 & \text{for } j \in D \end{array}$$

Online Facility Location Algorithm

Algorithm 5: OFL Algorithm

Input: (G, d, f, F)
$F^a \leftarrow \emptyset$; $D \leftarrow \emptyset$;
while *a new client j' arrives* **do**
  increase $\alpha_{j'}$ until one of the following happens:
  (a) $\alpha_{j'} = d(j', i)$ for some $i \in F^a$; /* connect only */
  (b) $f(i) = (\alpha_{j'} - d(j', i)) + \sum_{j \in D}(d(j, F^a) - d(j, i))^+$ for some $i \in F \setminus F^a$; /* open and connect */
  $F^a \leftarrow F^a \cup \{i\}$; $D \leftarrow D \cup \{j'\}$; $a(j') \leftarrow i$;
return (F^a, a);

Online Facility Location Results

The OFL has competitive ratio $\Theta\left(\frac{\log n}{\log \log n}\right)$ [Fotakis 2008]

There are randomized and deterministic $\mathrm{O}(\log n)$-competitive algorithms known for it [Meyerson 2001, Fotakis 2007]

[Nagarajan and Williamson 2013] give a dual-fitting analysis for the algorithm by [Fotakis 2007]

Acknowledgements

Thank you!

Questions?